On Vladimir Markov type inequality in $L^p$ norms on the interval $[-1; 1]$

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**Abstract**

We prove inequality $\|P^{(k)}\|_{L^p[-1, 1]} \leq B_p T_n^{(k)}(1) n^{k} \|P\|_{L^p[-1, 1]}$; where $B_p$ are constants independent of $n = \deg P$ with $1 \leq p \leq 2$, which is sharp in the case $k \geq 3$. A method presented in this note is based on a factorization of linear operator of $k$-th derivative throughout normed spaces of polynomial equipped with a Wiener type norm.

**Key words:** Vladimir Markov type inequality, $L^p$ norms

1. **Introduction.**

Consider a normed space $(\mathcal{P}(\mathbb{C}), \| \cdot \|)$ of polynomials of one variable equipped with a norm $\| \cdot \|$. The classical Vladimir Markov inequality (cf. [8],[16]) is the following inequality for $k$-th derivative of a polynomial $P$ of degree $n$

$$\|P^{(k)}\|_{[-1, 1]} \leq T_n^{(k)}(1) \|P\|_{[-1, 1]} = \frac{n^2(n^2 - 1) \cdots (n^2 - (k - 1)^2)}{1 \cdot 3 \cdots (2k - 1)} \|P\|_{[-1, 1]}$$

$$\leq C^k \frac{n^{2k}}{k!} \|P\|_{[-1, 1]}$$

(1.1)

with an absolute constant $C$. The meaning and its importance of the condition

$$\|P^{(k)}\| \leq C^k \frac{(\deg P)^{km}}{(k!)^m} \|P\|$$

was discovered in [2]. Grzegorz Sroka in his paper [20], motivated by [1] has obtained the inequality

$$\|P^{(k)}\|_{L^p[-1, 1]} \leq (C_k p + 1) k^2 \|T_n^{(k)}\|_{[-1, 1]} \|P\|_{L^p[-1, 1]},$$

where constants $C_k$ are bounded and $T_j$ are Chebyshev polynomials of the first kind (he showed that $C_k \leq \frac{12}{\pi^2} e^2$ for $k \geq 3$). As a corollary he derived the inequality of V. Markov’s type

$$\|P^{(k)}\|_{L^p[-1, 1]} \leq B_k \frac{1}{k!} n^{2k} \|P\|_{L^p[-1, 1]}.$$

Let us note that looking for optimal bounds of a type

$$\|P^{(k)}\|_{p_1} \leq C_n(k, p_1, p_2) \|P\|_{p_2}, \quad n = \deg P$$

is a subject of many investigations (cf. [12], [19], [7]).
2. Vladimir Markov’s type inequality

The main result of this note is the following improvement of [20] in the case $1 \leq p \leq 2$ (our arguments are quite different that used in [20]).

**Theorem 2.1.** If $1 \leq p \leq 2$, then for any polynomial $P$ of degree $k \leq \deg P \leq n$ we have inequalities

$$
||P^{(k)}||_p \leq B_p \max_{k \leq l \leq n} ||T^{(k)}_l||_p n^{2l} ||P||_p = B_p ||T^{(k)}_n||_p n^{2} ||P||_p,
$$

where

$$
||P||_p = \left( \int_{-1}^{1} |P(x)|^p dx \right)^{1/p}, \quad B_p = (3e/\pi)^{1/p}(2p + 2)^{1/p}.
$$

Here $T_n$ are the classical Chebyshev’s polynomials of the first kind.

As a non-obvious corollary we obtain a version of V. Markov’s property (it is a consequence of a fact that derivatives of $T_n$ are related to other Jacobi polynomials). It was discussed in [4], mainly in the case $p = 2$.

**Corollary 2.2.** For a fixed $1 \leq p \leq 2$ there exists a constant $C_p$ such that for all $k \geq 3$ we have Vladimir Markov’s type inequality

$$
||P^{(k)}||_p \leq C_p \frac{1}{k!} n^{2k} ||P||_p.
$$

**Remark 2.3.** The corollary is also true in the case $k = 1, 2$ but can not be derived from Theorem 2.1 (cf. remarks in [1] related to Z. Ciesielski results from [10] who investigated the behavior of $||T_n||_p$). In the case $k = 2$ and $1 < p \leq 2$ we can get a bound as in the corollary but with much worse constants.

In the proof of Theorem 2.1 we shall need the following important result. Let $x = \cos t = \frac{1}{2}(e^{it} + e^{-it})$ be element in the Wiener algebra of an absolute convergent trigonometric series $x = \sum_{n=-\infty}^{\infty} a_n e^{int}$ equipped with the $l^1$ Wiener norm $w_1(x) = \sum_{k=-\infty}^{\infty} |a_k|$.

Let $X_N = (P_N, w_1(P(x)))$, $B^N(x) = \{P \in P_N : w_1(P(x)) \leq 1\}$, where $P_N = \{P \in P(C) : \deg P \leq N\}$.

**Proposition 2.4.** (Baran, Milówka, Ozorka [5]) For an arbitrary $N \in \mathbb{N}$

$$
\text{extr}(B^N(x)) = \{\eta_0 T_0, \ldots, \eta_N T_N : |\eta_j| = 1, \ j = 0, \ldots, N\}.
$$

Here $\text{extr}(B^N(x))$ is the set of extreme points of the ball $B^N(x)$ (cf. [17] for this very classical notion and its importance), $T_j$ is $j$-th Chebyshev polynomial of the first kind.
Corollary 2.5. If $L$ is a linear operator on $\mathcal{P}_N$ then its norm between $(\mathcal{P}_N, w_1(P(x)))$ and $(\mathcal{P}_N, \| \cdot \|_p)$ is equal to $\max_{0 \leq j \leq N} \|LT\|_p$ that means $\|LP\|_p \leq \max_{0 \leq j \leq N} \|LT\|_p w_1(P(x))$ for $P \in \mathcal{P}_N$.

Now we shall prove Theorem 2.1.

Proof. Let $P(\cos t) = \sum_{j=-n}^{n} c_j e^{jt}$. We have, by the Hölder inequality,

$$
\sum_{j=-n}^{n} |c_j| \leq (2n + 1)^{1/p} \left( \sum_{j=-n}^{n} |c_j|^q \right)^{1/q}
$$

and applying the Hausdorff-Young inequality (c.f. [6],[22], which is a consequence of interpolation properties of spaces $L^p$), we shall get

$$
\sum_{j=-n}^{n} |c_j| \leq (2n + 1)^{1/p} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |P(\cos t)|^p dt \right)^{1/p}.
$$

Now we shall use the inequality like [13] (Lemma 3.1, p. 733)

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} |P(\cos t)|^p dt \leq 2np(1 + 1/(np))^{np+1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |P(\cos t)|^p |\sin t| dt,
$$

which gives

$$
w_1(P(\cos t)) = \sum_{j=-n}^{n} |c_j|
$$

$$
\leq (2n + 1)^{1/p}(2p + 1/n)^{1/p}(1 + 1/(np))^{n^{1/p}} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |P(\cos t)|^p |\sin t| \right)^{1/p}
\leq A_p n^{2/p} \|P\|_p
$$

with $A_p = (3e)^{1/p}(2p + 1)^{1/p}$.

Now the crucial step is to apply Corollary 2.5, which gives the most important bound

$$
\|P^{(k)}\|_p \leq \max_{k \leq l \leq n} \|T_l^{(k)}\|_p w_1(P(\cos t)).
$$

Applying the bound for $w_1(P(\cos t))$ we finish the proof:

$$
\|P^{(k)}\|_p \leq \max_{k \leq l \leq n} \|T_l^{(k)}\|_p B^p n^{2/p} \|P\|_p
$$

with $B_p = A_p^{1/p}$.

Remark 2.6. Let us note the following surprising fact, which can be observed in the proof above: a bound of the form $w_1(P(\cos t)) \leq B_p n^{2/p} \|P\|_p$ is analogous to the bound (Nikolski type inequality) $\|P\|_{[-1,1]} \leq C_p n^{2/p} \|P\|_p$, but $w_1(P(\cos t))$ can not be estimated by a product of a constant and $\|P\|_{[-1,1]}$. 

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REFERENCES


